

Dual Time-Stepping

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1 Governing Equations

Consider 1-D unsteady convection with a source term:

\[ Q_t + F_x = S \]  \ (1)

We can always modify this to consider a modified 1-D unsteady convection with a source and term that also contains a fictitious pseudo time, \( \tau \), in order to introduce the concept of dual time-stepping:

\[ Q_\tau + Q_t + F_x = S \]  \ (2)

2 Dual Time-Stepping

2.1 Euler Implicit

Now what happens if we apply dual time-stepping w/ 1st order backwards in time (Euler Implicit):

\[ \frac{Q^{p+1} - Q^p}{\Delta \tau} + \frac{Q^{p+1} - Q^n}{\Delta t} + F_x^{p+1} = S^{p+1} \]  \ (3)

It should be noted that in the subiteration process in pseudo time, we set \( Q^p = Q^n \) and when we have finished the process we set \( Q^{n+1} = Q^{p+1} \). Putting the equation in delta form \( \Delta Q^p = Q^{p+1} - Q^p \) and linearizing we have:

\[ \frac{\Delta Q^p}{\Delta \tau} = - \left[ \frac{\Delta Q^p + Q^p - Q^n}{\Delta t} + F_x^p + (F_Q \Delta Q)_x^p - S^p - (S_Q \Delta Q)^p \right] \]  \ (4)

Multiplying by \( \Delta t \) and grouping together the \( \Delta Q^p \) terms on the left-hand-side (LHS) we can write:

\[ \left( 1 + \frac{\Delta t}{\Delta \tau} \right) \Delta Q^p + \Delta t (F_Q \Delta Q)_x^p - \Delta t (S_Q \Delta Q)^p = -\Delta t \left[ \frac{Q^p - Q^n}{\Delta t} + F_x^p - S^p \right] \]  \ (5)

Finally, we can simplify to get unity in front of the first term:

\[ \Delta Q^p + \left( \frac{\Delta t}{1 + \frac{\Delta t}{\Delta \tau}} \right) (F_Q \Delta Q)_x^p - \left( \frac{\Delta t}{1 + \frac{\Delta t}{\Delta \tau}} \right) (S_Q \Delta Q)^p = -\Delta t \left[ \frac{Q^p - Q^n}{\Delta t} + F_x^p - S^p \right] \]  \ (6)

where, clearly:

\[ \frac{Q^p - Q^n}{\Delta t} + F_x^p - S^p \]  \ (7)

is the Residual.

If we let \( \Delta \tau \to \infty \) then:

\[ \Delta Q^p + \Delta t (F_Q \Delta Q)_x^p - \Delta t (S_Q \Delta Q)^p = -\Delta t \left[ \frac{Q^p - Q^n}{\Delta t} + F_x^p - S^p \right] \]  \ (8)
which is Newton-like subiteration.

Furthermore, if we don’t do any subiterations, then we have the traditional Euler implicit method:

\[
\Delta Q^n + \Delta t (F_Q \Delta Q)_x^n - \Delta t (S_Q \Delta Q)_x^n = -\Delta t \left[ F_x^n - S^n \right]
\]  
(9)

### 2.2 2nd Order Backwards

Similarly, we can apply dual time-stepping w/ 2nd order backwards in time (BDF2):

\[
\frac{Q^{p+1} - Q^p}{\Delta \tau} + \frac{3Q^{p+1} - 4Q^n + Q^{n-1}}{2\Delta t} + F_{x}^{p+1} = S^{p+1}
\]  
(10)

Putting this in delta form and linearizing we have:

\[
\frac{\Delta Q^p}{\Delta \tau} = - \left[ \frac{3\Delta Q^p + 3Q^p - 4Q^n + Q^{n-1}}{2\Delta t} + F_{x}^p + (F_Q \Delta Q)_x^p - S^p - (S_Q \Delta Q)_x^p \right]
\]  
(11)

Finally, we arrive at the following:

\[
\Delta Q^p + \left( \frac{2\Delta t}{1 + \frac{2\Delta \tau}{3\Delta x}} \right) (F_Q \Delta Q)_x^p - \left( \frac{2\Delta t}{1 + \frac{2\Delta \tau}{3\Delta x}} \right) (S_Q \Delta Q)_x^p = - \left( \frac{2\Delta t}{1 + \frac{2\Delta \tau}{3\Delta x}} \right) \left[ \frac{3Q^p - 4Q^n + Q^{n-1}}{2\Delta t} + F_{x}^p - S^p \right]
\]  
(12)

where, clearly:

\[
\frac{3Q^p - 4Q^n + Q^{n-1}}{2\Delta t} + F_{x}^p - S^p
\]  
(13)

is the Residual.

In this case, if we let \( \Delta \tau \to \infty \) then:

\[
\Delta Q^p + \frac{2}{3} \Delta t (F_Q \Delta Q)_x^p - \frac{2}{3} \Delta t (S_Q \Delta Q)_x^p = - \frac{2}{3} \Delta t \left[ \frac{3Q^p - 4Q^n + Q^{n-1}}{2\Delta t} + F_{x}^p - S^p \right]
\]  
(14)

which is Newton-like subiteration.

Furthermore, if we don’t do any subiterations, then we have the traditional 2nd order backwards method:

\[
\Delta Q^n + \frac{2}{3} \Delta t (F_Q \Delta Q)_x^n - \frac{2}{3} \Delta t (S_Q \Delta Q)_x^n = - \frac{2}{3} \Delta t \left[ - \frac{Q^n - Q^{n-1}}{2\Delta t} + F_{x}^n - S^n \right]
\]  
(15)

### 2.3 Trapezoidal

Similarly, we can apply dual time-stepping w/ Trapezoidal time marching:

\[
\frac{Q^{p+1} - Q^p}{\Delta \tau} + \frac{Q^{p+1} - Q^n}{\Delta t} + \frac{F_{x}^n + F_{x}^{p+1}}{2} = \frac{S^n + S^{p+1}}{2}
\]  
(16)

Putting this in delta form and linearizing we have:

\[
\frac{\Delta Q^p}{\Delta \tau} = - \left[ \frac{\Delta Q^p + Q^p - Q^n}{\Delta t} + \frac{F_{x}^p + F_{x}^p}{2} + (F_Q \Delta Q)_x^p - S^n + S^p + (S_Q \Delta Q)_x^p \right]
\]  
(17)

Finally, we arrive at the following:

\[
\Delta Q^p + \left( \frac{1}{2} \frac{\Delta t}{1 + \frac{\Delta \tau}{3\Delta x}} \right) (F_Q \Delta Q)_x^p - \left( \frac{1}{2} \frac{\Delta t}{1 + \frac{\Delta \tau}{3\Delta x}} \right) (S_Q \Delta Q)_x^p = - \left( \frac{1}{2} \frac{\Delta t}{1 + \frac{\Delta \tau}{3\Delta x}} \right) \left[ \frac{Q^p - Q^n}{\Delta t} + \frac{F_{x}^n + F_{x}^p}{2} - \frac{S^n + S^p}{2} \right]
\]  
(18)

where, clearly:

\[
\frac{Q^p - Q^n}{\Delta t} + \frac{F_{x}^n + F_{x}^p}{2} - \frac{S^n + S^p}{2}
\]  
(19)
is the Residual.

In this case, if we let \( \Delta \tau \to \infty \) then:

\[
\Delta Q^p + \frac{1}{2} \Delta t (F_Q \Delta Q)_x^p - \frac{1}{2} \Delta t (S_Q \Delta Q)^p = -\frac{1}{2} \Delta t \left[ \frac{Q^p - Q^n}{\Delta t} + \frac{F^n_x + F^p_x}{2} - \frac{S^n + S^p}{2} \right]
\]

which is Newton-like subiteration.

Furthermore, if we don’t do any subiterations, then we have the traditional trapezoidal method:

\[
\Delta Q^n + \frac{1}{2} \Delta t (F_Q \Delta Q)_x^n - \frac{1}{2} \Delta t (S_Q \Delta Q)^n = -\Delta t [F^n_x - S^n]
\]